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RECURRENCE FOR NON-ZERO RATIONAL PARAMETERS”
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ABSTRACT. In the present paper, we prove that self-approximation of $\log \zeta(s)$ with $d = 0$ is equivalent to the Riemann Hypothesis. Next, we show self-approximation of $\log \zeta(s)$ with respect to all nonzero real numbers d . Moreover, we partially filled a gap existing in ‘The strong recurrence for non-zero rational parameters’ and prove self-approximation of $\zeta(s)$ for $0 \neq d = a/b$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$.

1. INTRODUCTION

In 1981 Bagchi [1] discovered that the following almost periodicity holds in the critical strip if and only if the Riemann Hypothesis is true. To state it, let $\mu\{A\}$ stand for the Lebesgue measure of a measurable set A , $D := \{s \in \mathbb{C} : 1/2 < \operatorname{Re}(s) < 1\}$ and $H(K)$ denote the space of non-vanishing continuous functions on a compact set K , which are analytic in the interior, equipped with the supremum norm $\|\cdot\|_K$. Then Bagchi’s result can be formulated as follows (see also [6, Section 8]).

Theorem A. *The Riemann Hypothesis holds if and only if, for every compact set $K \subset D$ with connected complement and for every $\varepsilon > 0$, we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \|\zeta(s + i\tau) - \zeta(s)\|_K < \varepsilon \} > 0. \quad (1)$$

In 2010 Nakamura [3] showed the following property which might be called *self-approximation* of the Riemann zeta function.

Theorem B. *For every algebraic irrational number $d \in \mathbb{R}$, every compact set $K \subset D$ with connected complement and every $\varepsilon > 0$, we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \|\zeta(s + id\tau) - \zeta(s + i\tau)\|_K < \varepsilon \} > 0.$$

Note that the self-approximation with respect to almost all real numbers d was also verified in [3]. Afterwards, Pańkowski [5] showed the above result for any irrational number d whereas Garunkštis [2] and Nakamura [4] investigated the self-approximation for non-zero rational numbers, independently. Unfortunately, the papers [2] and [4] contain a gap in the proof of the main theorem, so actually their methods work only for the logarithm of the Riemann zeta function (see Remark 2.3).

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In this paper, we prove that self-approximation of $\log \zeta(s)$ with $d = 0$ is equivalent to the Riemann Hypothesis in Theorem 2.1. Next, we show self-approximation of $\log \zeta(s)$ for all nonzero real numbers d in Theorem 2.2. Moreover, we partially filled the gap mentioned above and prove self-approximation of $\zeta(s)$ for $0 \neq d = a/b$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$ in Theorem 3.1.

2. SELF-APPROXIMATION OF $\log \zeta(s)$

Firstly, we show the following theorem which is an analogue of Theorem A.

Theorem 2.1. *The Riemann Hypothesis holds if and only if, for every compact set $K \subset D$ with connected complement and for every $\varepsilon > 0$, we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \|\log \zeta(s + i\tau) - \log \zeta(s)\|_K < \varepsilon \} > 0. \quad (2)$$

Proof. If the Riemann hypothesis is true we can apply Voronin's universality theorem.

Suppose that there exists a zero $\xi \in D$ of $\zeta(s)$. Put $K_\varepsilon := \{s \in \mathbb{C} : |s - \xi| \leq \varepsilon\} \subset D$. Now assume that for a neighborhood K_ε of ξ the following relation holds:

$$\|\log \zeta(s + i\tau) - \log \zeta(s)\|_{K_\varepsilon} < \varepsilon. \quad (3)$$

If a zero ρ of $\zeta(s)$, where $\rho \in K_\varepsilon(\tau) := \{s \in \mathbb{C} : |s - \xi - i\tau| \leq \varepsilon\}$ does not exist, then the function $\log \zeta(s + i\tau)$ is analytic in the interior of K_ε and bounded on K_ε . This contradicts to the above inequality. Hence a zero ρ of $\zeta(s)$ exists in $K_\varepsilon(\tau)$. With regard to (3) and the definition of $K_\varepsilon(\tau)$ the zeros ξ and ρ are intimately related; more precisely, $|\rho - \xi - i\tau| < \varepsilon$. Thus two different shifts τ_1 and τ_2 can lead to the same zero ρ , but their distance is bounded by $|\tau_1 - \tau_2| < 2\varepsilon$. Therefore we obtain this lemma by modifying the proof of [6, Theorem 8.3] and using the classical Rouché theorem. \square

The reasoning of [5, Theorem 1.1] and [2, Theorem 1] can be easily applied to prove self-approximation of $\log \zeta$.

Theorem 2.2. *For every real number $d \neq 0$, every compact set $K \subset D$ with connected complement and for every $\varepsilon > 0$, it holds that*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \|\log \zeta(s + i\tau) - \log \zeta(s + id\tau)\|_K < \varepsilon \} > 0. \quad (4)$$

Remark 2.3. In fact, in [2] and [4] it was claimed that the above theorem holds even for the Riemann zeta function instead of the logarithm of $\zeta(s)$. Unfortunately, the proofs of main results in [2, Theorem 1] and [4, Corollary 1.2] are not sufficient, since it was only shown that $|\zeta(s + di\tau)/\zeta(s + i\tau) - 1| < \varepsilon$. Obviously we have

$$|\zeta(s + i\tau) - \zeta(s + id\tau)| = |\zeta(s + i\tau)| |\zeta(s + id\tau)/\zeta(s + i\tau) - 1|.$$

So in order to prove self-approximation of $\zeta(s)$ it should have been proved that $\zeta(s + i\tau)$ is not too large, namely we need the inequality $|\zeta(s + i\tau)| < |\zeta(s + id\tau)/\zeta(s + i\tau) - 1|^{-1}$ which is not necessarily satisfied.

3. SELF-APPROXIMATION OF $\zeta(s)$

In the following theorem we partially fix the gap existing in [2, Theorem 1] and [4, Corollary 1.2] and prove self-approximation of $\zeta(s)$ in the case $0 \neq d = a/b \in \mathbb{Q}$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$.

Theorem 3.1. *For every $0 \neq d = a/b$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$, every compact set $K \subset D$ and for every $\varepsilon > 0$, we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \|\zeta(s + i\tau) - \zeta(s + id\tau)\|_K < \varepsilon \} > 0. \quad (5)$$

Proof. First of all, note that it suffices to consider the case $a \neq b$, or equivalently $d \neq 1$.

Let us take $\omega(p_m) := \exp(2\pi i m/(a - b))$, where p_m denotes the m -th prime number. Then we have $\omega^a(p) = \omega^b(p)$ since $\omega^{a-b}(p) = 1$. Firstly, we show that

$$|\zeta_z(s, \omega^c)| \leq \exp \left(\frac{7(1 - z^{1-2\sigma})}{2\sigma - 1} + |a - b|(z^{-\sigma} + 2|s|(1 - z^{-\sigma})) \right), \quad (6)$$

where $c = a, b$ and $\zeta_z(s, \omega^c) := \prod_{p \leq z} (1 - \omega(p)^c p^{-s})^{-1}$. In order to prove it let us consider the function $-\sum_{p \leq z} \log(1 - \omega(p)^c p^{-s})$. Then

$$-\sum_{p \leq z} \log \left(1 - \frac{\omega(p)^c}{p^s} \right) = \sum_{p \leq z} \sum_{k=1}^{\infty} \frac{\omega(p)^{ck}}{k p^{ks}} = \sum_{p \leq z} \frac{\omega(p)^c}{p^s} + \sum_{p \leq z} \sum_{2 \leq k} \frac{\omega(p)^{ck}}{k p^{ks}}.$$

Let us estimate the latter sum on the right hand side. For $\sigma > 1/2$ one has

$$\begin{aligned} \left| \sum_{p \leq z} \sum_{2 \leq k} \frac{\omega(p)^{ck}}{k p^{ks}} \right| &\leq \sum_{p \leq z} \sum_{2 \leq k} \frac{1}{p^{k\sigma}} = \sum_{p \leq z} \frac{1}{p^{2\sigma} - p^{\sigma}} \leq 7 \sum_{p \leq z} \frac{1}{p^{2\sigma}} \\ &\leq 7 \sum_{2 \leq n \leq z} \frac{1}{n^{2\sigma}} \leq 7 \int_1^z t^{-2\sigma} dt = \frac{7(1 - z^{1-2\sigma})}{2\sigma - 1}. \end{aligned}$$

To consider the former sum, we put

$$\omega(n) := \begin{cases} \omega(p) & \text{if } n = p \leq z, \\ 0 & \text{otherwise,} \end{cases} \quad \Omega_z := \sum_{n=1}^z \omega(n)^c.$$

Then, by partial summation, we obtain

$$\sum_{p \leq z} \frac{\omega(p)^c}{p^s} = \sum_{n=1}^z \frac{\omega(n)^c}{n^s} = \frac{\Omega_z}{z^s} - \sum_{n=1}^{z-1} \Omega_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s} \right) = \frac{\Omega_z}{z^s} + s \sum_{n=1}^{z-1} \Omega_n \int_n^{n+1} t^{-s-1} dt.$$

Let us notice that $\omega(p)^c$ is a nontrivial root of unity, since $\gcd(a, b) = 1$ implies $a - b \nmid a, b$. Hence we have

$$\left| \sum_{p \leq z} \frac{\omega(p)^c}{p^s} \right| \ll z^{-\sigma} + |s| \int_1^z t^{-\sigma-1} dt = z^{-\sigma} + \frac{|s|}{\sigma} (1 - z^{-\sigma}) \leq \frac{1}{z^{\sigma}} + 2|s| \left(1 - \frac{1}{z^{\sigma}} \right),$$

where the constant in symbol \ll is equal to $|a - b|$. Therefore we obtain (6).

Now it suffices to follow the steps of the proof of [2, Theorem 1] and the following fact. Let $\|x\|$ give the the distance from a real number x to the the nearest integer. Then the set of positive real numbers τ satisfying

$$\max_{p_m \leq z} \left\| \tau \frac{\log p_m}{2\pi} - \frac{m}{a-b} \right\| < \delta$$

has a positive density for every positive δ by the Kronecker approximation theorem. Thus it holds for sufficiently large z

$$\|\log \zeta(s + ic\tau) - \log \zeta_z(s, \omega^c)\|_K < \varepsilon, \quad c = a, b.$$

This completes the proof. □

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